Fair valuation of path-dependent participating life insurance contracts

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Received November 2002; received in revised form 3 May 2003; accepted 25 August 2003

Abstract

Fair valuation of insurance contracts, and of options embedded in them, is an important, incompletely understood issue. With the coming IAS insurance contract standard, the valuation of liabilities in life insurance is due to a drastic change. We present a computationally tractable model for fair valuation of participating life insurance contracts with given, almost arbitrary bonus policies. Unlike traditional valuation methods, our model captures several essential features of participating life insurance contracts, such as fair values of interest rate guarantees and of various bonus policies. In the model, fair value of life insurance contracts is understood as the arbitrage free price in the presence of liquid markets for liabilities. In addition to numerical results, the model gives solutions in closed form.

JEL classification: G13; G22; G23
MSC: IM10; IE10

Keywords: Life insurance; Fair valuation; Participating contracts; Embedded options; Bonus policy

1. Introduction

Insurance accounting is experiencing a radical shift from traditional valuation of insurance liabilities towards fair valuation of liabilities. This has occurred partly because one side on the balance sheet is valued in market values—the assets—while the other—the liabilities—is not. Another important reason for the change of emphasis on accounting is the need for a better understanding of financial risks, such as interest rate guarantees, and, in general, of valuation of various elements in insurance contracts. This is especially timely now, since low interest rates have made interest rate guarantees an important issue for life insurance contracts. Several insurance companies, such as Nissan Mutual Life, have run into trouble partially as a result of an underestimation of embedded options in their written insurance contracts.
To address these issues, the forthcoming IAS standard for insurance contracts has adopted a radical way of valuation: insurance liabilities are to be valued as if they were traded in huge numbers among well-informed, independent investors in a liquid marketplace. This is called *fair valuation*, which is formally defined as “the amount for which an asset could be exchanged or a liability settled between knowledgeable, willing parties in an arm’s length transaction”; see IASB (2001). According to the draft IAS standard, liabilities including embedded options, should be valued with stochastically estimated future cash flows and discounted with riskless interest, which is often interpreted as interest of 30-year government bonds.

Although International Accounting Standards Board (IASB) seems to be moving away from demanding use of strict fair valuation, importance of understanding fair valuation is not reduced, since the original motivation for the valuation method—reliable valuation of embedded options and indirect obligations—still remains.

To better understand how fair valuation is feasible, we construct a model for valuation of participating life insurance contracts, by extending the work of Grosen and Jørgensen (2000), and by deriving an analytical solution for the fair value of the contract. For certain simple bonus mechanisms our model produces nice analytic results, but for most bonus policies we get iterated integral representations that yield results only after relying on numerics. Furthermore, the formulation of the model allows easy incorporation of various kinds of bonus mechanisms, thus leading to a more practical and comprehensive approach to the understanding of the value of participating contracts. We shall also briefly consider how to include a known term structure of riskless interest to the model.

It is well known that option valuation models capture well some aspects of market’s valuation process, hence we use them as a starting point. We take it as given that a fair value is the arbitrage free price\(^1\) for assets and liabilities, as in option valuation models. We assume that liabilities are valued as if a liquid market for the underlying contracts existed.

2. Background on life insurance contracts

Before going on to the actual modeling, let us first briefly review the primary characteristics of the contracts. The value of a *unit-linked contract* is directly linked to the performance of a portfolio of assets (the unit) associated with the contract. Such contracts may include maturity guarantees that can be interpreted as options giving a right to a certain minimal amount at the end of the contract period. Valuation of such contracts is explained, e.g. in Ekern and Persson (1996).

In *participating contracts* the policyholder is entitled to certain part of the profits generated by the assets associated with the contract. The division of profits between the policyholder, a reserve, etc. is dictated by the bonus policy. The bonus policy often includes some form of guaranteed interest protecting the investor from turbulence of financial markets. This type of contracts are also known as with-profits contracts.

Contracts may also offer the policyholder an opportunity to choose between several bonus mechanisms in the middle of the contract term. The right to switch from one bonus mechanism to another is an embedded option, and it is natural to ask what is the fair value of such right. We call such provisions *switch options*.

Insurance contracts can be called *European* or *American* in the same vein as in the regular option literature, with the tacit understanding that the terms refer to the surrender option. Presence (corr. absence) of surrender option in a contract simply means that the policyholder can (resp. cannot) discontinue the contract before the term. A contract with a surrender option is called American, a contract without a surrender option is called European.

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\(^1\) We do not include a market value margin—the margin the market demands for bearing the risk inherent in the contract, see IASB (2001)—into our model.
3. Valuation of participating contracts

3.1. Contracts and assets

Following Grosen and Jørgensen (2000), we examine a participating life insurance contract, into which the policyholder invests an initial amount at \( t = 0 \). If the policyholder lives for \( T \) years, he or she gains a payoff which consists of invested capital, of guaranteed interest, and of bonuses. In case of death of the policyholder before maturity of the contract, a mortality benefit is paid out, and the contract is terminated.

Let us first analyze the contract as a purely financial instrument, and we shall later explain how to take the mortality effects into account in Section 6. The following are the notations for the relevant parameters of the contract: \( T \) is the length of the contract, \( I_0 \) the initial investment, \( A_0 \) the initial investment in a risky asset, \( P_0 \) the initial payoff value, \( P_G \) the guaranteed payoff at maturity and \( r_G \) the guaranteed interest as an annual interest.

Policyholder makes an investment of \( I_0 \) into the policy. Only part of the initial investment, \( A_0 \), is invested, and the policy has initially a payoff value \( P_0 \). Typically, \( I_0 > A_0 \), since the initial investment includes various fees and setup costs. Depending on the payoff rules, the initial payoff value may or may not equal the initial value of the investment into equity. At maturity, the payoff \( P_T \) is fully paid to the policyholder.

The variables describing the value of the contract are as follows: \( A(t) \) the value of the associated assets at time \( t \) and \( P(t) \) the payoff value of the policy at time \( t \), with the initial conditions \( A(0) = A_0 \) and \( P(0) = P_0 \). In general, values of \( A(t) \) and \( P(t) \) do not coincide, even if \( A_0 = P_0 \). This will be necessary, as the contract will typically involve income guarantees that have to be compensated by dividing only a portion of the profits from the investments to the payoff value. The difference \( A(t) - P(t) \) determines how much excess assets the policy has at a time \( t \), and we shall call it the bonus reserve—the bonus reserve can then also be negative.

We consider here a particular, but also quite typical, type of evolution for the payoff value of the contract: every time a year has elapsed, the payoff value will accumulate a guaranteed interest and possible bonuses. Our choice of year as the time interval is quite arbitrary, and any other choice of intervals could also be analyzed similarly. The exact value of the annual interest depends on how the investments have evolved, and we assume the annual interest is determined by a bonus policy which was fixed at the time of making of the contract. We shall discuss the definition of bonus policies separately in Section 3.2.

We shall also inspect contracts which can be terminated before maturity. As in the usual option terminology, a contract will be called European, if it can only be surrendered at maturity. With a mixed American contract, we shall mean a contract which allows exercising the surrender option at every full year, i.e. immediately after the interest has been granted, but not at any other time during the year (hence the term mixed American). For simplicity, we also assume that there are no penalties for an early termination of the contract, i.e. that the entire payoff value is always paid to the policyholder when the contract is terminated.

3.2. The bonus policy

We are interested in contracts which have an annual interest whose value depends on the evolution of the investment in the market. In particular, we shall inspect here contracts where the annual interest depends only on how long the contract has been in effect, on the current value of the policy, and on the value of the investment at present and at the granting of the previous interest. In a mathematical notation, the evolution of the payoff value, \( P(t) \), during the year \( v \in \{1, 2, \ldots, T\} \), i.e. from \( t = v - 1 \) to \( t = v \), is then given by

\[
P(t) = \begin{cases} 
P(v - 1), & \text{when } t \in [v - 1, v), \\
B_v(A(v), A(v - 1), P(v - 1)) & \text{at } t = v.
\end{cases}
\]

(1)

The actual bonus function \( B_v \) used can be different for each year \( v \), but we assume that each \( B_v \) depends only on the present value of the investment \( A(v) \), the value of the investment a year before \( A(v - 1) \), and the previous...
payoff value \( P(v - 1) \). Especially, we allow the family of bonus functions \( \{ B_v \}_{v=1}^T \) to take into account the value of any predetermined asset, such as the riskless bond, by allowing the use of a different \( B_v \) for each year.

We also assume that each \( B_v \) is deterministic, in the sense that it is determined at the time of making of the contract, and that it is a continuous function of its parameters. As we do not let the policy ever to lose all of its value, we also assume \( B_v(a', a, p) > 0 \) for all \( a', a, p > 0 \).

The most common example of this kind of bonus policy is a contract which has a guaranteed minimum annual interest \( r_G \), but which grants a higher interest \( R \) in a year when the investments have been doing well. Assuming the bonus policy is kept fixed, these policies can be described by setting all the functions \( B_v \) equal to

\[
B(A(v), A(v - 1), P(v - 1)) = \left[ 1 + \max\{r_G, R(A(v), A(v - 1), P(v - 1))\} \right] P(v - 1).
\] (2)

Such contracts have already been considered in the literature before, for instance, in Jensen et al. (2001) and Bacinello (2001). The model of Jensen et al. assumes that part of the profits is granted as bonuses to the policyholder, while another part is taken out to maintain a bonus reserve from where the interest can be paid out in a bad year. Their model is, in the present notation, equivalent to using the bonus interest function

\[
R(A(v), A(v - 1), P(v - 1)) = \alpha \frac{A(v - 1)}{A(v)} - 1 - \gamma,
\] (3)

where the parameters satisfy \( 0 \leq \alpha \leq 1 \) and \( \gamma \geq 0 \).

The bonus policy considered by Bacinello would amount to using the bonus interest function

\[
R(A(v), A(v - 1), P(v - 1)) = \alpha \left( \frac{A(v)}{A(v - 1)} - 1 \right),
\] (4)

where \( 0 \leq \alpha \leq 1 \). This bonus policy is of the unit-linked type, since it only depends on the performance of the assets \( A(t) \) during the year and not on the previous payoff value of the policy.

A contract with an additional guaranteed minimum payoff \( P_G \) at maturity would then be described by using the bonus function \( B \) in (2) for \( B_v \) when \( v = 1, \ldots, T - 1 \), and defining for \( v = T \),

\[
B_T(A(T), A(T - 1), P(T - 1)) = \max\{P_G, \overline{B}(A(T), A(T - 1), P(T - 1))\}.
\] (5)

A consequence of this fairly general choice of bonus policy is that evaluation of fair value of an American contract, which allows the exercise of the surrender option at any moment, becomes prohibitively difficult. Although the above bonus policies certainly include policies for which it is never optimal to exercise the surrender option during the year, this is not true in general. For instance, assuming that the bonus policy does not include any guaranteed interest, the valuation of the surrender option essentially reduces to valuation of a put option on shares with discrete dividends. Valuation of such contracts is a hard problem, see, e.g. Karatzas and Shreve (1998). Therefore, we restrict the study here to surrender options which can only be exercised at full years.

4. Stochastic model

In this section, we construct a stochastic model for valuation of life insurance contracts based on Grosen and Jørgensen (2000) and Jensen et al. (2001). Here we inspect the more general bonus mechanism introduced in Section 3.2, and we employ a different analytical approach to solving the model, as well as more accurate numerical methods. We will aim at the modeling of a contract which is of the participating type and which includes a surrender option with guaranteed interest rate and a guaranteed payoff at maturity.

The main goal is to measure the expected liability \( V(t) \) of a contract of the type defined in Section 3. Liability \( V(t) \) measures the nominal present value of the policy at time \( t \), taking into account the expected future cash flows, and we shall define it as the fair value of the policy. In particular, \( V(0) \) then represents the fair value of the (at the moment purely financial) contract at the time of its making.
The difference between $P$ and $V$ may not be obvious: $P(t)$ describes how much the policy is worth to the policyholder at time $t$ if the policyholder exercises the surrender option. The value of $V(t)$ describes the fair value of the policy, i.e. the prospectively computed value that describes how much future cash flows caused by the contract would be worth today. While $P(t)$ is known definitely at time $t$, liability $V(t)$ is an estimate that is computed as an expected present value of the future cash flows.

We take it as given that the fair value is the arbitrage free price of such a contract. As in the Black–Scholes model, the arbitrage free price of a life insurance contract is computed as an expected value with respect to a risk-neutral measure, see, e.g. Musiela and Rutkowski (1997). Although participating life insurance contracts have more structure than plain options in the Black–Scholes model, the end result of the modeling will be that the valuation of participating contracts with bonuses is very similar to studying options on stocks paying discrete dividends.

We assume that the market consists of our tradable contract $P$ and of two other investment opportunities: the earlier mentioned equity $A$ and a riskless bond $L$. The time evolution of $A$ and $L$ we assume to be determined by the stochastic differential equations

$$
\begin{align*}
&dL = rdL, \\
&dA = \mu A dt + \sigma A dW
\end{align*}
$$

meaning that $A$ follows the usual geometric Brownian motion. The evolution of the payoff value $P(t)$ is then determined by (1), and we shall inspect the evolution of its market value in the next section.

The parameters in (6) are as follows: $r$ is the interest rate of the riskless bond $L$, $\mu$ the average growth of $A$ and $\sigma$ the volatility of $A$. Both the interest rate and the volatility are assumed to be deterministic and constant, although our methods could be easily accommodated to the non-constant deterministic case (we shall comment on how to do this later in Section 5.1). It is also worth emphasizing that $r$ is a continuous interest rate, whereas $r_{ij}$ is an annual interest, and thus $e^{r_{ij}} - 1$ would be an annual interest comparable to $r_{ij}$. The volatility $\sigma$ should be understood as an implied volatility, see, e.g. Musiela and Rutkowski (1997).

### 4.1. Fair value of a contract

The fair value of the contract at maturity is equal to its payoff value, i.e. we have a terminal condition $V(T) = P(T)$. Since we assume that the contract can be sold freely on the market, its fair value at other times is determined by the risk-neutral measure of the arbitrage free market of $(L, A, P)$. with the natural filtration $\mathcal{F}_t$ the fair value of a European contract is for $t < T$ given by the random variable

$$
V^\mathbb{Q}(t) = E^\mathbb{Q}[e^{-r(T-t)}P(T)|\mathcal{F}_t],
$$

where $E^\mathbb{Q}$ refers to an expectation value in the risk-neutral measure.

The fair value at time $t$ is thus determined by variables which define the future evolution of $P(t)$ uniquely.

For the types of contracts based on the bonus policies presented in Section 3.2, it suffices to use the triplet $(A(t), A([t]), P([t]))$ as the history (the notation $[t]$ refers to the integer part of $t$, i.e. to an integer $n$ for which $n \leq t < n + 1$). We can thus write for all $0 \leq t \leq T$,

$$
V^\mathbb{Q}(t) = V(t, A([t]), P([t])),
$$

and we shall call a function $V(t, a', a, p)$, which is defined for $0 \leq t \leq T$ and for all $a', a, p > 0$, and which satisfies (8) almost surely, a European fair value function. Note that if $t \in \{0, 1, \ldots, T\}$, then $A(t) = A([t])$ and only the value $V(t, a, a, p)$ is ever used in the relation (8) at integer times $t$.

Fair value of an American contract is obtained similarly, with the exception that the contract may terminate before maturity $T$.

$$
V^A(t) = \sup_{s \in \mathcal{T}_s,t} E^\mathbb{Q}[e^{-r(T-t)}P(T)|\mathcal{F}_s],
$$

where $\mathcal{T}_s,t$ is the class of $\mathcal{F}_s$ stopping times taking values in $[t, T]$—for details, see, e.g. Karatzas and Shreve (1998).
This valuation problem is, however, hard to solve which is the main reason why we restricted the study to the mixed American contracts defined in Section 3. Eq. (9) is also valid for mixed American contracts if we define that the stopping times in for all $\alpha, p > 0$ by
\[ V^\alpha(t) = V(t, A(t), A(t), P(t)), \]
for all $\alpha, p > 0$ and for almost every $t < v$. We shall first assume that the values of both the American and the European contracts have to be almost surely left continuous functions at $v$, i.e.
\[ V^E(t) \to V^E(v) \] and $V^\alpha(t) \to V^\alpha(v)$ almost surely when $t \to v^-$, as otherwise we have an arbitrage opportunity.

To see this, let $V$ be either $V^E$ or $V^\alpha$, and consider a realization of $A(t)$ for which $V(t) \to V(v)$. Then there is $\varepsilon > 0$ and a sequence of times $t_n < v$ such that $t_n \to v$, and either $V(t_n) \geq V(v) + \varepsilon$ for all $n$, or $\tilde{V}(t_n) \geq V(v) + \varepsilon$ for all $n$. In the first case, one can make arbitrage by buying the contract at a time $t_0$ which is sufficiently close to $v$ and then selling the contract at $t = v$. In the second case, one first sells the contract short at a suitable $t_n$, and then buys it back at $t = v$. The arbitrage argument relies on our assumption that $B_t$ is a predetermined continuous function of $A(t)$ and $A(t)$ is continuous almost surely, since then the value of $\tilde{V}(v)$, will become asymptotically certain when $t \to v^-$. In the European case, we therefore obtain that almost surely
\[ \lim_{t \to v^-} V(t, A(t), A(v-1), P(v-1)) = V(v, A(v), A(v), P(v)) = F_\alpha(A(v), B_v(A(v), A(v-1), P(v-1))), \]
for all $\alpha, p > 0$ and for almost every $t$ in the essential range of $P(v-1)$ the following "no-jump" condition needs to hold:
\[ \lim_{t \to v^-} V(t, A', A, P) = F_\alpha(A', B_v(A', A, P)). \]

Suppose next that we know the value function for a time $t_0 \in (v-1, v)$. Then for any fixed $A(v-1)$ and $P(v-1)$, the finding of a fair value function on the interval $(v-1, t_0)$ has reduced exactly to a Black–Scholes valuation problem. Therefore, it follows from the work of Black and Scholes (1973) that the value function $V$ satisfies the Black–Scholes equations in this interval. More precisely, we now know that there is a function $f : (-\infty, t_0) \times \mathbb{R}_+ \to \mathbb{R}$ which is differentiable and satisfies for all $r < t_0$ the equation
\[ \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + r x f - df = 0, \]
where $\sigma$ is the volatility of the underlying asset and $r$ is the risk-free interest rate. The solution to this equation will be the value function $V$.
and using which we can obtain a fair value function for times \( t \in [v-1, v] \) by defining \( V(t, x, A(v-1), P(v-1)) = f(t, x) \) for all \( x > 0 \). As the solution to the Black–Scholes equations is unique, the definition of \( V(t, \cdot, \cdot) \) will then be independent of which \( t_0 > t \) we use here.

Taking the limit \( t_0 \to v \) in the above and applying Eq. (13) yields a recipe for the computation of the fair value for any \( v-1 \leq t < v \) when the value at \( t = v \) is known. First fix \( a = A(v-1) \) and \( p = P(v-1) \) to be any allowed values. Then solve the Black–Scholes equation (14) with the boundary condition

\[
f(v, x) = F(x, B(v, a, p)),
\]

and define \( \tilde{V}(t, a', a, p) = f(t, a') \) for all \( v-1 \leq t < v \) and \( a' > 0 \). By our results, the function so defined will be a European fair value function in this time interval.

Since the value is known at maturity, we now have an algorithm for the solution of the fair value of a European contract: start at time \( t = T \) with the function \( F_T(a, p) = p \) for all \( a, p > 0 \). Then apply the above procedure to find the fair value function \( V(t, \cdot, \cdot) \) at \( t = T-1 \), and use it to define \( F_{T-1} \). Iterate the procedure until \( F_0 \) has been computed. The fair value of the contract is then equal to \( F_0(A_0, P_0) \).

### 4.2.2. Mixed American contract

Similar methods can be applied in our mixed American case. To see this, let \( v \in \{1, \ldots, T\} \) and consider a \( t \in (v-1, v) \). As then \( T_{v,T} = T_{v,v} \), and for any stopping time \( \tau \in T_{v,v} \), we have the identity \( \mathbb{E}^q[\mathbb{E}^v[e^{-r(\tau-t)}P(\tau)|F_t]] = \mathbb{E}^v[e^{-r(\tau-t)}P(\tau)|F_t] \), we can easily see then that

\[
\hat{V}^v(t) = \mathbb{E}^v[e^{-r(T-t)}\hat{V}^{v+1}(t)|F_t].
\]  

(16)

This implies that if we know the value function \( \hat{V} \) at \( t=v \), we can use the European methods to solve its value at any \( v-1 < t < v \).

Without going through the proof, we then propose that there exists a sequence of functions \( \tilde{F}_v(a, p) \), defined for all \( a, p > 0 \) and \( v = 0, 1, \ldots, T \), such that the stopping time

\[
\tau_v = \min\{v \in \mathbb{N} \cap [v, T-1] | \tilde{F}_v(A(v), P(v)) = P(v) \} \cup \{T\}
\]  

(17)

is optimal, i.e. yields the supremum in (9).

We furthermore propose that the functions \( \tilde{F}_v \) can be obtained by iterating the following procedure: start by defining \( \tilde{F}_T(a, p) = p \) for all \( a, p > 0 \). Then, for any \( v \in [v, T-1] \) and any \( a, p > 0 \), compute the solution \( f(t, x) \) to the Black–Scholes equation (14) with the terminal boundary condition \( f(v, x) = \tilde{F}_v(A(v), B(v, a, p)) \), and define the next function \( \tilde{F}_{v-1} \) by

\[
\tilde{F}_{v-1}(a, p) = \max_{a'} \{f(v-1, a', a, p)\}.
\]

(18)

With the above optimal stopping time, we can also conclude that almost surely for any \( v \in [0, 1, \ldots, T] \),

\[
\hat{V}^v(t) = \tilde{F}_v(A(v), P(v))
\]

(19)

and the solution to the mixed American valuation problem is therefore given by \( \tilde{F}_0(A_0, P_0) \).

Note that the above stopping time is a formalization of the following intuitively obvious rule: the owner of the policy at \( t = v-1 \) will decide to cash out the policy if the fair value of a corresponding European contract, i.e. of a contract which lasts for 1 year and has a payoff function \( \hat{V}(v, A(v), A(v), P(v)) \), would be less than the present payoff value \( P(v-1) \). The fact that we can use a European type fair value between the years follows from Eq. (16).

### 4.2.3. Contract with a switch option

Let us next assume that the policyholder is annually, at the beginning of each year \( v \) immediately after the bonuses have been granted, given \( n \) different bonus policies \( \{B^k\}_{k=1}^n \) to choose from. The iterative method presented above can also be applied to this problem, with only a small change in the iteration step.
Let \( \tilde{F}_v \) be a fair value function at time \( v \) of a contract with the above switch option, and consider any \( a, p > 0 \).

For the iteration step of a European contract with a switch option, first solve the Black–Scholes equation with the following boundary conditions for the functions \( f_k \):

\[
 f_k(v, x) = \tilde{F}_v(x, B_k^v(x, a, p)),
\]

and then define the next boundary function, instead of Eq. (15), by

\[
 \tilde{F}_{v-1}(a, p) = \max_k \{ f_k(v - 1, a) \}.
\]

The rationale behind the above method is that a rational policyholder would always choose the bonus policy leading to the highest fair value.

Similarly, for a mixed American contract with a switch option, define the next boundary function, instead of Eq. (18), by

\[
 \tilde{F}_{v-1}(a, p) = \max \{ p, \max_k \{ f_k(v - 1, a) \} \},
\]

where \( f_k \) are defined as in the previous paragraph. Only one payoff value needs to be considered in Eq. (22), since the choice of the bonus policy for each year is made after the bonuses from the previous year are granted.

We would like to stress once more that in all of the previous computations we have relied on the assumption that it is possible to buy and to sell liabilities freely on a liquid marketplace.

5. Computation of the fair value

5.1. Analytical solution

There are several standard methods available for the solution of the Black–Scholes equation (14), and the only additional ingredient we have here is the no-jump condition. In fact, our valuation problem is similar to solving the Black–Scholes equations for options on stocks which give discrete dividends, see, e.g. Björk (1998). In our case, however, bonuses (corresponding to dividends) do not diminish the payoff value of the contracts (corresponding to stocks).

The key step in the iterative procedure we defined in the previous section was a solution of the Black–Scholes equations with a given terminal condition. In this section, we shall derive an integral representation of the solution to the proposed iteration step, i.e. we shall derive a function \( f(t, x) \) which is differentiable and satisfies Eq. (14) for all \( t < v \) and \( x > 0 \), which is left continuous at \( t = v \), and which satisfies the terminal boundary condition \( f(v, \cdot) = f_0(\cdot) \) with a given function \( f_0 : \mathbb{R}^+ \to \mathbb{R} \).

Suppose first that \( f(t, x) \) is such a solution. With the aid of the auxiliary parameters \( \omega = r/\sigma^2 - 1/2 \) and \( \rho = (1/2)(\sigma^2 t^2 + r) \), we define the function \( g : (-\infty, v] \times \mathbb{R} \to \mathbb{R} \) by the formula

\[
 f(t, x) = x^{-\omega} e^{\sigma^2 t/2} \left( \frac{1}{\sigma} \ln x \right).
\]

Then, for all \( y \in \mathbb{R} \), \( \lim_{t \to v^-} g(t, y) = g_0(y) \), where \( g_0(y) = e^{\sigma^2 y - \rho y} f_0(e^{\sigma y}) \), and, for all \( t < v \) and \( y \in \mathbb{R} \), the function \( g \) satisfies a backward diffusion equation

\[
 \frac{\partial g}{\partial t} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 g}{\partial y^2} = 0.
\]

Given a terminal boundary function \( g_0 \), Eq. (24) has a unique solution which is obtained from a convolution of \( g_0 \) with a Gaussian distribution. The regularity requirements on the boundary function are very weak: for instance, if \( g_0 \) is locally integrable and exponentially bounded, then the convolution yields the unique solution to the problem.
This solution can be transformed, by a suitable change of variables in the convolution, to a solution of the original problem. This way we find that for any given, sufficiently regular (e.g. continuous and polynomially bounded) terminal condition \( f_0 \), the function \( f \) defined for \( x > 0 \) and \( t < v \) by

\[
f(t,x) = e^{-\rho(v-t)} \int_0^\infty \frac{1}{\sqrt{2\pi(v-t)\sigma^2}} e^{-\left(\ln \lambda \right)^2/(2v-t)\sigma^2} \lambda^{-1} f(\lambda x) \, d\lambda,
\]

is a solution to the Black–Scholes equation with the required boundary conditions.

In our iteration scheme, \( f_0 \) is determined by the “no-jump” condition (13). Applying this to (25) and performing another change of variables given by \( \lambda(u) = \exp(\sigma u + r - \left(1/2\right)\sigma^2) \), we get the iteration step for the European fair value function \( F_v \) from year \( v \) to year \( v - 1 \),

\[
F_{v-1}(a,p) = e^{-r} \int_{-\infty}^{\infty} G(u) F_v(\lambda(u)a, B_v(\lambda(u)a, a, p)) \, du,
\]

where \( G \) is the density of the standard normal distribution,

\[
G(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}.
\]

The same computations can be applied to the American case, and we obtain the iteration step for the American fair value function \( \bar{F}_v \) from Eq. (18) yielding, again with \( \lambda(u) = \exp(au + r - \left(1/2\right)\sigma^2) \),

\[
\bar{F}_{v-1}(a,p) = \max \left\{ p, e^{-r} \int_{-\infty}^{\infty} G(u) \bar{F}_v(\lambda(u)a, B_v(\lambda(u)a, a, p)) \, du \right\}.
\]

In fact, the above formulae easily generalize to a case when \( r \) and \( \sigma \) have a deterministic time dependence. The only change needed would be to replace \( r \) and \( \sigma^2 \) by their cumulative time averages, i.e. to change

\[
r \rightarrow \frac{1}{v-t} \int_t^v r(s) \, ds \quad \text{and} \quad \sigma^2 \rightarrow \frac{1}{v-t} \int_t^v \sigma^2(s) \, ds
\]

in the above. However, these generalizations will not be needed in the following examples, and we shall not discuss them any further.

5.2. Numerics

We have presented a solution to the problem of computation of the fair value as an iteration procedure, and we derived an explicit integral representation for the iteration step in Section 5.1. Now we need to discuss the evaluation of the iteration step in practice. Apart from some simple examples, an analytical solution of the integrals is not in general possible. The iteration step for the fair value of the contract could be computed directly by numerically integrating the Black–Scholes equation, but the above integral representation should allow more control over the accuracy of the approximations involved.

We start our scheme at maturity with the terminal condition giving initial values to the lattice. As we showed earlier, further values of the liability are then computed with the iterative solution of the functions \( F_v \) in the European case and \( \bar{F}_v \) in the American case, going backwards by yearly steps starting from maturity. In both cases, the key missing step is the evaluation of the following integral for all \( a, p > 0 \):

\[
K(a,p) = e^{-r} \int_{-\infty}^{\infty} G(u) F(\lambda(u)a, B(u)\lambda(u)a, a, p) \, du,
\]

where \( \lambda(u) = \exp(au + r - \left(1/2\right)\sigma^2) \), \( G \) is defined in (27) and the function \( F(\cdot, \cdot) \) is assumed to be known. For any chosen values of \( a \) and \( p \) the evaluation could be done to a very high accuracy but the real problem is to find an approximation to the entire function.
The simplest method, which we have also used in the examples presented in Section 7, is to represent the function \( F_v \) (or \( \bar{F}_v \)) by computing and storing its values on a sufficiently fine discrete lattice, and then by employing an interpolation scheme to compute the values between the lattice sites. For the interpolation, we used a simple piecewise linear approximation.\(^2\)

Given the values of \( F_v \) stored on an appropriate lattice, we need to choose a lattice on which we approximate the function \( F_v-1 \). This lattice need not to be the same as the lattice on which the function \( F_v \) has been evaluated. The simplest choice is a square lattice consisting of points \((a_i, p_j)\) with \( i = 1, \ldots, I \) and \( j = 1, \ldots, J \), i.e. defining \( a_i = a_{\text{max}}/I \) and \( p_j = p_{\text{max}}/J \) for some given \( I, J, a_{\text{max}} \) and \( p_{\text{max}} \). During the simulation we need to store only two lattices, one for the values of \( F_v \) and the other for the values of \( F_v-1 \).

We need to compute the integral (30) at every lattice site in order to get the approximation for the function \( F_v - 1 \). The integral we computed by employing a piecewise linear approximation of the function \( x \mapsto F(x, B_v(x, a, p)) \), and we shall conclude the section by deriving the result from this approximation.

Let \((a, p)\) be a given lattice site and let \( A_{\text{max}} > 0 \) denote the maximum value of \( a \) for which we have stored \( F(a', p) \). Divide the interval \([0, A_{\text{max}}]\) into \( M \) pieces, i.e. define for all \( k = 0, 1, \ldots, M-1 \)

\[
x_k = \frac{A_{\text{max}}}{M} k,
\]

and evaluate\(^3\) \( \phi_k = F(x_k, B_v(x_k, a, p)) \). The determination of a reasonable value for \( M \) depends on the bonus policy \( B_v \). We shall use a linear approximation away from these points to define the integrand, i.e. we use in (30) the approximation

\[
F(x, B_v(x, a, p)) \approx \sum_{k=0}^{M-1} \left( b_{k,0} x + b_{k,1} \right) \chi_{[k, k+1)}(x) + b_{M,1} x + b_{M,0} \chi_{(A_{\text{max}}, \infty)}(x),
\]

where \( \chi_{\omega} \) is the characteristic function of the set \( \omega \) and the constants \( b_k \) are defined for all \( k = 0, \ldots, M-1 \) by

\[
b_{k,0} = \phi_k - k(\phi_{k+1} - \phi_k), \quad b_{k,1} = \frac{M}{A_{\text{max}}} (\phi_{k+1} - \phi_k),
\]

and for \( k = M \) by

\[
b_{M,0} = b_{M-1,0}, \quad b_{M,1} = b_{M-1,1}.
\]

Depending on \( B_v \), some other extrapolation method may be needed instead of the linear extrapolation term \((b_{M,1} x + b_{M,0}) \chi_{(A_{\text{max}}, \infty)}(x)) \) in Eq. (32) to keep numerical errors better under control.

We next note that the evaluation of the integral (30) can then be reduced to the evaluation of the values of the function

\[
\hat{\theta}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy,
\]

which can be computed easily using the error function found in most standard numerical libraries. To see this, first apply the linear approximation (32) to (30) when, by using the properties of function \( \hat{\theta} \), we get

\[
K(a, p) = \sum_{k=0}^{M} \left( b_{k,0} \int_{x_k}^{x_{k+1}} G(u) \, du + b_{k,1} e^{-r} \int_{x_k}^{x_{k+1}} G(u) \, du \right),
\]

\(^2\) In fact, we started out by trying a naive spline approximation instead of the linear one, but this, paradoxically, reduced the accuracy of the results. This was, at least partially, due to the form of the spline approximation, as it allowed having \( F_v(a, p) < F_v(a', p) \) even if \( a > a' \).

\(^3\) We assumed in the above that the limit value \( \phi_0 = \lim_{x \to 0^+} F(x, B_v(x, a, p)) \) can be computed.
where we have defined $u_0 = -\infty$, $u_{M+1} = \infty$ and for $k = 1, \ldots, M$

$$u_k = \frac{1}{\sigma} \ln \left( \frac{M}{a} \right) - \frac{r}{\sigma} + \frac{1}{2} \beta.$$  \hfill (37)

Since for any $\alpha$ and $\beta$, $\int_\beta^\alpha G(u) \, du = \Theta(\alpha) - \Theta(\beta)$, we arrive—after a brief reshuffling of the terms of the sum, and using $\Theta(\infty) = 0$—at the formula

$$K(a, p) \approx M \sum_{k=0}^{M} \left[ a(b_{k,1} - b_{k-1,1})\Theta(u_k - \sigma) + e^{-r}(b_{k,0} - b_{k-1,0})\Theta(u_k) \right],$$  \hfill (38)

where we have defined $b_{-1,1} = 0$. For numerical evaluation of the $k = 0$ term in the sum, we also note that $\Theta(-\infty) = 1$. To improve performance of the computations, it was useful to evaluate and store the needed values of the function $\Theta$ before the first iteration step. In addition, in the limit $a \to 0^+$, the values of integral (30) can typically be computed analytically, which may be helpful in computations on a lattice.

6. Life insurance with mortality

So far we have only considered purely financial contracts without mortality, studying in a sense life insurance without mortality. Now we wish to study more realistic life insurance contracts, which means inclusion of mortality and a mortality benefit into the model. We assume, as is common in the insurance mathematics literature, that the mortality risk is diversifiable. If the policyholder lives for $T$ years, he or she gains a payoff which consists of invested capital, of guaranteed interest, and of bonuses. In case of death of the policyholder before maturity of the contract, only a mortality benefit $K$ is paid out, and the contract is terminated.

In our model, the liability of a European contract after $t$ years is given by

$$V_k(t) = T - tp_{x+t} + tV(t) + KA_{x+t},$$  \hfill (39)

where $V$ is the liability without mortality, as calculated previously, $x$ the age of the policyholder at the beginning of the contract, and $p_{x+t}$ the probability of a $x + t$ years old person to reach an age of $x + T$ years. The last term $A$ on the right-hand side is the nominal present value of the unit mortality benefit, see Bowers et al. (1997) for the precise definition and for a derivation of (39).

7. Example computations

In this section, we present results from a few example calculations in which we compared how some of the earlier discussed features of the participating contracts affect the fair value of the contract. The definition and computation of the fair value we have explained in the previous sections, and here we apply these methods to a contract of the type inspected by Jensen et al. (2001): for all years we use the bonus function $B_{v} = B$, where $B$ is defined by Eqs. (2) and (3), and we assume that the initial payoff value is equal to the initial investment, i.e. that $P_0 = A_0$.

The bonus function has two parameters which define how much ($\gamma$) of the profits is retained in the bonus reserve, and how much ($\alpha$) is given out as bonuses to the policies. How these and the various other parameters were chosen, will be explained separately for each of the examples below.

All values of the contracts presented in the tables are fair values at the beginning of the contract. As required by the no-arbitrage condition, the fair values were discounted with the riskless interest. Numerical fair values were stored on the same square $(a, p)$-lattice at all times, and the lattice had $100 \times 100$ points and the maximum value at $(100000, 200000)$. For the numerical integration we used $A_{\text{max}} = 100000$ and $M = 100$. 

Table 1
The fair value of a participating contract, without mortality, for several different riskless interests

<table>
<thead>
<tr>
<th>Continuous riskless interest</th>
<th>2%</th>
<th>4%</th>
<th>6%</th>
</tr>
</thead>
<tbody>
<tr>
<td>American contract</td>
<td>16689</td>
<td>13329</td>
<td>12129</td>
</tr>
<tr>
<td>European contract</td>
<td>16689</td>
<td>13027</td>
<td>10788</td>
</tr>
<tr>
<td>Riskless bond</td>
<td>13338</td>
<td>8941</td>
<td>5993</td>
</tr>
<tr>
<td>Bonus option</td>
<td>3351</td>
<td>4086</td>
<td>4795</td>
</tr>
<tr>
<td>Surrender option</td>
<td>0</td>
<td>302</td>
<td>1341</td>
</tr>
</tbody>
</table>

See the text for a description of the terminology and of the other parameters relevant to the contract.

7.1. Participating contract

The fair value of a European contract can be divided into two parts: a riskless bond and a bonus option. The value of the riskless bond is the guaranteed payoff discounted to present. The value of the bonus option in a European contract is defined as the fair value of the contract minus the bond value, whereas the value of the surrender option is the fair value of an American contract minus that of a European contract.

In the following computations, the initial investment is 10 000, the volatility $\sigma = 15\%$, and the guaranteed interest $r_G = 3.5\%$. The length of the contract is taken to be 20 years.

7.1.1. Participating contract without mortality

In Table 1, we show the computed fair values for several different choices of the continuous riskless interest. The results were obtained by using the earlier explained simulations with the parameter values $\alpha = 0.85$ and $\gamma = 0.1$. In the table, “American” and “European” refer to a contract of a mixed American and of a European type, respectively. The value of the riskless bond is the discounted value of guaranteed income, $e^{-rT}(1 + r_G)^T$. When riskless interest is low, all of the values of the contract come from guaranteed interest, so that fair values of mixed American and European contracts coincide. When riskless interest is higher, the value of alternative investments will be higher as well, and the fair value of the surrender option gets larger. This is reflected in the fair value of the mixed American contract which in this case becomes larger than the fair value of the corresponding European contract.

The value of the riskless bond element is lower for higher riskless interest, which is a result of riskless interest $r$ dominating $r_G$ for high values of $r$. On the other hand, the expected yield from the risky investment will be higher for a higher riskless interest. Still, more bonuses do not compensate here for the lowering of the bond value with higher riskless interests.

The reason for the phenomenon, that the higher the riskless interest, the lower the total fair value of the contract, is that with higher riskless interest the lower value of the bond element dominates over the growth in bonus options.

7.1.2. Participating contract with mortality

For an inclusion of the effect of mortality, let us consider a policyholder who is a 40-year-old male. We use the mortality tables used in the Finnish statutory employment pension system in 2001. By using the fixed mortality benefit of 15 000, we get the results presented in Table 2. The values shown are for a European contract, with the same bonus policy as in the previous section.

Note that $r_G = 3.5\%$ annually is more than a continuous interest $r = 2\%$. 
Table 2
The fair value of a participating contract, including mortality effects

<table>
<thead>
<tr>
<th>Continuous riskless interest</th>
<th>2%</th>
<th>4%</th>
<th>6%</th>
</tr>
</thead>
<tbody>
<tr>
<td>European contract</td>
<td>16323</td>
<td>12747</td>
<td>10535</td>
</tr>
<tr>
<td>Provision for life</td>
<td>15473</td>
<td>12078</td>
<td>10002</td>
</tr>
<tr>
<td>Provision for death</td>
<td>850</td>
<td>669</td>
<td>533</td>
</tr>
<tr>
<td>Life</td>
<td>12366</td>
<td>8289</td>
<td>5556</td>
</tr>
<tr>
<td>Riskless bond</td>
<td>3107</td>
<td>3789</td>
<td>4446</td>
</tr>
</tbody>
</table>

* The terminology used here, and the parameters of the contract, are explained in the text.

This time the contract is first divided into two parts: provision for life and provision for death. The provision for life has been further divided into a riskless bond and a bonus option. The value of the riskless bond is now computed as \( 10000 \times (1 + r_G)^{20} \times e^{-20r} \times 20p_{40} \), where \( 20p_{40} \) is the probability that a 40-year-old Finnish male survives until the age of 60.

It is worth noting that the contract with mortality benefit is not the same as the contract without the benefit plus an option element that consists of the mortality benefit. The inclusion of the mortality benefit here changes the structure of the contract, hence the values of Tables 1 and 2 are not directly comparable.

Table 2 shows behavior similar to that shown in Table 1: the fair value of the riskless bond diminishes as the riskless interest gets higher, and, as before, the value of the bonus option is higher for a higher riskless interest. The diminution of the provision for death for a higher riskless interest results from keeping the mortality benefit fixed. In summary, although the contract is different from the one studied in the previous section, the overall behavior is quite similar.

7.2. Contract with a switch option

We next study a participating contract which includes a right to switch between multiple portfolios having different investment profiles and bonus policies. The additional element here is the policyholder’s right to choose, annually and without additional costs, a bonus policy from a predetermined collection of policies in the beginning of each year. Under these circumstances, a rational policyholder will always switch to the portfolio that has the highest risk-neutral value at the beginning of the year under study. The value of the switch option is then the value of the whole contract with the option to switch minus the value of the most valuable portfolio studied alone. The values of other options, as well as the rest of the terminology, are the same as in the previous section.

The length of the contract is 20 years, and it includes a right to switch between two portfolios on the fly. Portfolio 1 has a guaranteed annual interest of 2.5% and volatility of 30%, and it uses the bonus policy determined by \( \alpha = 0.5 \) and \( \gamma = 0.2 \). Portfolio 2 has a guaranteed annual interest of 4%, volatility of 10% and employs \( \alpha = 0.25 \) and \( \gamma = 0.1 \). Portfolio 1 is riskier than portfolio 2, which retains higher guaranteed interest. The value of the riskless bond is now \( 10000 \times e^{-20r} \times 1.04^{20} \), where the higher of the guaranteed interests is chosen as the reference interest.

Neglecting the effect of mortality, we obtained the results presented in Table 3. Evidently, a right to switch has quite high a value. When the riskless interest is low, the option to switch has a higher value since neither portfolio is optimal for all sample paths. When the riskless interest is high, the higher bonuses from portfolio 1 dominate over the guaranteed interest rate in portfolio 2, but the right to switch still has considerable value.
Table 3
The fair value of a participating contract, without mortality effects, but with a right to switch between two different portfolios

<table>
<thead>
<tr>
<th>Continuous riskless interest</th>
<th>2%</th>
<th>4%</th>
<th>6%</th>
</tr>
</thead>
<tbody>
<tr>
<td>European contract</td>
<td>18943</td>
<td>14235</td>
<td>10988</td>
</tr>
<tr>
<td>European Riskless bond</td>
<td>14688</td>
<td>9845</td>
<td>6600</td>
</tr>
<tr>
<td>Bonus option</td>
<td>2355</td>
<td>3200</td>
<td>3538</td>
</tr>
<tr>
<td>Switch option</td>
<td>1900</td>
<td>1190</td>
<td>850</td>
</tr>
</tbody>
</table>

a The terminology, and the parameters of the portfolios, are explained in the text.

8. Discussion

The traditional method of accounting of insurance liabilities is very different from the method presented here. Traditionally, option-like elements embedded into insurance contracts have been taken into account implicitly using prudently estimated discount factors, without explicit estimation of the value of each embedded option. To gain better understanding of the financial risks associated with insurance contracts, one has to estimate the value of embedded options explicitly.

We developed a model for the estimation of the fair value of path-dependent, participating life insurance contracts and option elements embedded in them, assuming that liabilities are valued as if they were traded on a liquid market. The model produces analytic expressions for contracts of very short duration, and gives an iterative method for evaluation of contracts of longer duration. As the bonus functions can be chosen fairly freely, our model adapts well to several different situations. For instance, we could use it to analyze how solvency requirements affect the fair valuation of life insurance policies.

Using numerics, our model yields explicit values for various kinds of contracts and their option elements. The model behaves numerically relatively well, although we do not believe that the numerical procedure we used in the examples would be optimal. One possible improvement could be to employ a logarithmic scale instead of the linear one for the $(a, p)$-lattice.

To the best of our knowledge, the model presented here has not really been used before, despite its remarkable simplicity. The model sheds light on the problem of fair valuation of path-dependent liabilities, and it offers a basis for the modeling of liabilities of several types of path-dependent, participating insurance contracts.

Acknowledgements

AT wishes to thank Jukka Rantala, Asko Tanskanen, Esko Kivisaari, Luis Alvarez, Jorma Leinonen, Jussi Pelkonen and Onerva Savilaisten for helpful comments, and Varma-Sampo (nowadays called Varma) for providing a stimulating environment. JL acknowledges the support from the Academy of Finland grant no. 200231, and wishes to thank the hospitality of the Department of Mathematics of University of Helsinki, where part of the work was accomplished. We would also like to thank the referee for careful reading of the manuscript and for suggestions for several improvements.

References


